

L^2 -BETTI NUMBERS OF HYPERSURFACE COMPLEMENTS

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ABSTRACT. In [DJL07] it was shown that if \mathcal{A} is an affine hyperplane arrangement in \mathbb{C}^n , then at most one of the L^2 -Betti numbers $b_i^{(2)}(\mathbb{C}^n \setminus \mathcal{A}, \text{id})$ is non-zero. In this note we prove an analogous statement for complements of complex affine hypersurfaces in general position at infinity. Furthermore, we recast and extend to this higher-dimensional setting results of [FLM09, LM06] about L^2 -Betti numbers of plane curve complements.

1. INTRODUCTION

Let M be any topological space and $\alpha : \pi_1(M) \rightarrow \Gamma$ an epimorphism to a group Γ (all groups are assumed countable). Then for $i \in \mathbb{N} \cup \{0\}$ we can consider the L^2 -Betti number $b_i^{(2)}(M, \alpha) \in [0, \infty]$. We recall the definition and some of the most important properties of L^2 -Betti numbers in Section 2.

Let $X \subset \mathbb{C}^n$ ($n \geq 2$) be a reduced affine hypersurface defined by a polynomial equation $f = f_1 \cdots f_s = 0$, where f_i are the irreducible factors of f . Denote by $X_i := \{f_i = 0\}$, $i = 1, \dots, s$, the irreducible components of X , and let

$$M_X := \mathbb{C}^n \setminus X$$

be the hypersurface complement. Then M_X has the homotopy type of a finite CW complex of dimension n . It is well-known that $H_1(M_X; \mathbb{Z})$ is a free abelian group generated by the meridian loops γ_i about the non-singular part of each irreducible component X_i of X . Throughout the paper we denote by ϕ the map $\pi_1(M_X) \rightarrow \mathbb{Z}$ given by sending each meridian γ_i to 1. This is the same map as the homomorphism $f_* : \pi_1(M_X) \rightarrow \pi_1(\mathbb{C}^*) = \mathbb{Z}$ induced by f . We also refer to ϕ as the *total linking number homomorphism*. We call an epimorphism $\alpha : \pi_1(M_X) \rightarrow \Gamma$ *admissible* if the total linking number homomorphism ϕ factors through α .

The main result of this note is the following “nonresonance-type” theorem.

Theorem 1.1. *Let $X \subset \mathbb{C}^n$ be a reduced affine hypersurface in general position at infinity, i.e., whose projective completion intersects the hyperplane at infinity transversely. If $\alpha : \pi_1(M_X) \rightarrow \Gamma$ is an admissible epimorphism, then the L^2 -Betti numbers of the complement M_X are computed by:*

$$b_i^{(2)}(M_X, \alpha) = \begin{cases} 0, & \text{for } i \neq n, \\ (-1)^n \chi(M_X), & \text{for } i = n, \end{cases}$$

where $\chi(M_X)$ denotes the Euler characteristic of M_X .

Date: April 11, 2012.

2000 Mathematics Subject Classification. Primary 14J70, 32S20; Secondary 46L10, 58J22.

Key words and phrases. L^2 -Betti numbers, hypersurface complement, singularities, infinite cyclic cover, Alexander invariants, Milnor fibration, higher-order degree.

The author was partially supported by NSF-1005338.

In particular,

$$(-1)^n \cdot \chi(M_X) \geq 0.$$

As an immediate consequence we note the following:

Corollary 1.2. *If $X \subset \mathbb{C}^n$ is a reduced affine hypersurface defined by a homogeneous polynomial (i.e., X is the affine cone on a reduced projective hypersurface in \mathbb{CP}^{n-1}), and $\alpha : \pi_1(M_X) \rightarrow \Gamma$ is an admissible epimorphism, then*

$$b_i^{(2)}(M_X, \alpha) = 0, \text{ for all } i \geq 0.$$

Indeed, it is easy to see that such an affine cone $X \subset \mathbb{C}^n$ is in general position at infinity. Moreover, by the existence of a *global Milnor fibration* [Mi68], the complement M_X is the total space of a fibration over \mathbb{C}^* , hence $\chi(M_X) = 0$.

In [DJL07] it was shown that if \mathcal{A} is an affine hyperplane arrangement in \mathbb{C}^n , then at most one of the L^2 -Betti numbers $b_i^{(2)}(\mathbb{C}^n \setminus \mathcal{A}, \text{id})$ is non-zero. Theorem 1.1 can be seen as an analogous statement for the complement of a hypersurface in \mathbb{C}^n which is in general position at infinity. In particular, we recast and generalize to arbitrary dimensions some results of [LM06, FLM09], where the case $n = 2$ of plane curve complements was considered.

In this note, we are also concerned with the L^2 -Betti numbers of the infinite cyclic cover defined by the total linking number homomorphism ϕ . More precisely, given an affine hypersurface $X \subset \mathbb{C}^n$, we denote by \widetilde{M}_X the infinite cyclic cover of M_X corresponding to ϕ . Moreover, for an admissible epimorphism $\alpha : \pi_1(M_X) \rightarrow \Gamma$ we let $\widetilde{\Gamma} := \text{Im}\{\pi_1(\widetilde{M}_X) \rightarrow \pi_1(M_X) \xrightarrow{\alpha} \Gamma\}$, and we denote the induced map $\pi_1(\widetilde{M}_X) \rightarrow \widetilde{\Gamma}$ by $\widetilde{\alpha}$. The L^2 -Betti numbers we are interested in are

$$b_i^{(2)}(\widetilde{M}_X, \widetilde{\alpha} : \pi_1(\widetilde{M}_X) \rightarrow \widetilde{\Gamma}).$$

In [Ma06], the author showed that for hypersurfaces $X \subset \mathbb{C}^n$ in general position at infinity the ordinary Betti numbers $b_i(\widetilde{M}_X)$ of the infinite cyclic cover \widetilde{M}_X are finite for all $0 \leq i \leq n-1$. In this note, we prove a non-commutative generalization of this fact. More precisely, we show the following:

Theorem 1.3. *Assume that the affine hypersurface $X \subset \mathbb{C}^n$ is in general position at infinity. Then the L^2 -Betti numbers $b_i^{(2)}(\widetilde{M}_X, \widetilde{\alpha})$ are finite for all $0 \leq i \leq n-1$.*

As it was already observed in several recent papers, e.g., see [DL06, DM07, Ma06], hypersurfaces in general position at infinity behave much like weighted homogeneous hypersurfaces up to homological degree $n-1$; see Prop.3.1 for a computation of L^2 -Betti numbers of weighted homogeneous hypersurface complements. The above Theorem 1.3 comes as a confirmation of this philosophy.

Another motivation for studying L^2 -Betti numbers of the infinite cyclic cover \widetilde{M}_X comes from the fact that for appropriate choices of the group Γ these numbers specialize into several classical Alexander-type invariants of the complement. For instance, following work of Cochran and Harvey (cf. [Co04, Ha05]), we can consider the following homomorphism

$$\pi_n : \pi_1(M_X) \rightarrow \pi_1(M_X) / \pi_1(M_X)_r^{(m+1)} =: \Gamma_m,$$

where given a group G we denote by $G_r^{(m)}$ the m -th term in the rational derived series of G . The group Γ_m is a poly-torsion-free-abelian (PTFA) group and we

define the *higher-order degrees* $\delta_{i,m}(X)$ of X as the dimension of the i -th homology of \widetilde{M}_X with coefficients in the skew field associated to $\widetilde{\Gamma}_m$. Similar invariants were defined and studied in [LM06, LM07] in the case of plane curves. Moreover, as noted in [FLM09], the higher-order degrees $\delta_{i,m}(X)$ can be regarded as L^2 -Betti numbers of the infinite cyclic cover \widetilde{M}_X , so the results of this note characterize the Cochran-Harvey invariants as well. At this point we want to emphasize that the higher-order degrees of a space M , hence also the L^2 -Betti numbers of the infinite cyclic cover \widetilde{M} , may as well be infinite, since \widetilde{M} is not in general a finite CW-complex. For example, for a topological space M with $\pi_1(M)$ a free group with at least two generators, the higher-order degrees $\delta_{1,m}$ are infinite (cf. [Ha05, Ex.8.2]). The finiteness results of this note are rigidity properties specific to the complex algebraic setting. We should also mention that if X is irreducible and in general position at infinity, then it is easy to see (cf. [LM06]) that for all $0 \leq i \leq n-1$ the integer $\delta_{i,0}(X)$ equals the degree of the i -th Alexander polynomial of X , or as shown in [Ma06], the i -th Betti numbers of the infinite cyclic cover \widetilde{M}_X . In the general reducible case, Libgober pointed out a nice relationship between $\delta_{i,0}(X)$ and the *support* of the i -th universal abelian Alexander module of the complement M_X (for more details, see [LM06] and the references therein).

Our result in Theorem 1.1 is reminiscent of a similar calculation by Jost-Zuo [JZ00] of L^2 -Betti numbers of a compact Kähler manifold of non-positive sectional curvature. This was considered in relation to an old question of Hopf whether the Euler characteristic of a compact manifold M of even real dimension $2n$ has sign equal to $(-1)^n$, provided M admits a metric of negative sectional curvature. However, the statement of our Theorem 1.1 is metric independent (and the degree of X is arbitrary). Finally, one should not be misled by these calculations into thinking that the L^2 -Betti numbers of finite CW-complexes are always integers, or that most of them usually vanish. In fact, the Atiyah conjecture asserts that these L^2 -Betti numbers are always rational; see [Lü02a, Lü02b] for more details on this conjecture and related matters.

This paper is organized as follows. In Section 2 we recall the definition of L^2 -Betti numbers and list some of their properties. We also express the Cochran-Harvey higher-order degrees of an affine hypersurface complement as L^2 -Betti numbers of the infinite cyclic cover of the complement. The main results, Theorem 1.1 and 1.3 are proved in Section 3.

Acknowledgement. The author would like to thank Anatoly Libgober and Stefan Friedl for sharing their comments on an earlier version of the paper.

2. L^2 -BETTI NUMBERS

2.1. The von Neumann algebra and its localizations. Let Γ be a countable group. Define

$$l^2(\Gamma) := \{f : \Gamma \rightarrow \mathbb{C} \mid \sum_{g \in \Gamma} |f(g)|^2 < \infty\}$$

the Hilbert space of square-summable functions on Γ . Then Γ acts on $l^2(\Gamma)$ by right multiplication, i.e., $(g \cdot f)(h) = f(hg)$. This defines an injective map $\mathbb{C}[\Gamma] \rightarrow \mathcal{B}(l^2(\Gamma))$, where $\mathcal{B}(l^2(\Gamma))$ is the set of bounded operators on $l^2(\Gamma)$. We henceforth view $\mathbb{C}[\Gamma]$ as a subset of $\mathcal{B}(l^2(\Gamma))$.

The *von Neumann algebra* $\mathcal{N}(\Gamma)$ of Γ is defined as the closure of $\mathbb{C}[\Gamma] \subset \mathcal{B}(l^2(\Gamma))$ with respect to pointwise convergence in $\mathcal{B}(l^2(\Gamma))$. Note that any $\mathcal{N}(\Gamma)$ -module \mathcal{M} has a dimension $\dim_{\mathcal{N}(\Gamma)}(\mathcal{M}) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$. We refer to [Lü02b, Def.6.20] for details.

2.2. L^2 -Betti numbers. Definition. Properties. Let M be a topological space (not necessarily compact) and let $\alpha : \pi_1(M) \rightarrow \Gamma$ be an epimorphism to a group. Denote by M_Γ the regular covering of M corresponding to α , and consider the $\mathcal{N}(\Gamma)$ -chain complex

$$C_*^{sing}(M_\Gamma) \otimes_{\mathbb{Z}[\Gamma]} \mathcal{N}(\Gamma),$$

where $C_*^{sing}(M_\Gamma)$ is the singular chain complex of M_Γ with right Γ -action given by covering translation, and Γ acts canonically on $\mathcal{N}(\Gamma)$ on the left. The i -th L^2 -Betti number of the pair (M, α) is defined as

$$b_i^{(2)}(M, \alpha) := \dim_{\mathcal{N}(\Gamma)}(H_i(C_*^{sing}(M_\Gamma) \otimes_{\mathbb{Z}[\Gamma]} \mathcal{N}(\Gamma))) \in [0, \infty].$$

We refer to [Lü02b, Def.6.50] for more details. Note that if M is a CW-complex of finite type, then the cellular chain complex $C_*(M_\Gamma)$ can be used in the above definition of L^2 -Betti numbers.

In the following lemma we summarize some of the properties of L^2 -Betti numbers. We refer to [Lü02b, Thm.6.54, Lem.6.53 and Thm.1.35] for the proofs.

Lemma 2.1. Let M be a topological space and let $\alpha : \pi_1(M) \rightarrow \Gamma$ be an epimorphism to a group.

- (1) $b_i^{(2)}(M, \alpha)$ is a homotopy invariant of the pair (M, α) .
- (2) $b_0^{(2)}(M, \alpha) = 0$ if Γ is infinite and $b_0^{(2)}(M, \alpha) = \frac{1}{|\Gamma|}$ if Γ is finite.
- (3) If M is a finite CW-complex, then

$$\sum_i (-1)^i b_i^{(2)}(M, \alpha) = \chi(M),$$

where $\chi(M)$ denotes the Euler characteristic of M .

Remark. For the definition of L^2 -Betti numbers of a pair (M, α) we do not need to require that the homomorphism α is surjective. However, we can reduce ourselves to this case since, for an arbitrary homomorphism $\alpha : \pi_1(M) \rightarrow \Gamma$, we have that

$$(1) \quad b_i^{(2)}(M, \alpha : \pi_1(M) \rightarrow \text{Im}(\alpha)) = b_i^{(2)}(M, \alpha : \pi_1(M) \rightarrow \Gamma).$$

A free Γ -CW complex \widetilde{M} is the same as a regular covering $p : \widetilde{M} \rightarrow M$ of a CW complex M with Γ as group of covering transformations. As a generalization of the homotopy invariance of L^2 -Betti numbers, we have the following result (see [Lü02b, Thm.1.35(1)]):

Lemma 2.2. Let $\tilde{f} : \widetilde{N} \rightarrow \widetilde{M}$ be a Γ -map of free Γ -CW complexes of finite type, and denote by $f : N \rightarrow M$ the induced map on the corresponding orbit spaces. If the homomorphism $H_i(\tilde{f}; \mathbb{C}) : H_i(\widetilde{N}; \mathbb{C}) \rightarrow H_i(\widetilde{M}; \mathbb{C})$ is bijective for $i \leq d-1$ and surjective for $i = d$, then:

$$(2) \quad b_i^{(2)}(M; \alpha : \pi_1(M) \rightarrow \Gamma) = b_i^{(2)}(N; \alpha \circ f_* : \pi_1(N) \xrightarrow{f_*} \pi_1(M) \xrightarrow{\alpha} \Gamma), \text{ for } i < d,$$

and

$$(3) \quad b_d^{(2)}(M; \alpha) \leq b_d^{(2)}(N; \alpha \circ f_*).$$

Finally, the L^2 -Betti numbers provide obstructions for a closed manifold to fiber over the circle S^1 . More precisely, by [Lü02b, Thm.1.39], we have the following:

Lemma 2.3. *Let M be a CW complex of finite type, and $f : M \rightarrow S^1$ a fibration with connected fiber F . Assume that the epimorphism $f_* : \pi_1(M) \rightarrow \pi_1(S^1) = \mathbb{Z}$ admits a factorization $\pi_1(M) \xrightarrow{\alpha} \Gamma \xrightarrow{\beta} \mathbb{Z}$, with α and β epimorphisms. Then*

$$b_i^{(2)}(M, \alpha) = 0, \quad \text{for all } i \geq 0.$$

2.3. L^2 -Betti numbers and Cochran-Harvey invariants. A group Γ is called *locally indicable* if for every finitely generated non-trivial subgroup $H \subset \Gamma$ there exists an epimorphism $H \rightarrow \mathbb{Z}$. In the following we refer to a locally indicable torsion-free amenable group as a LITFA group. We refer to [Lü02b, p.256] for the definition of an *amenable* group, but we note that any solvable group is amenable, while groups containing the free groups on two generators are not amenable. Also, note that a subgroup of a LITFA group is itself a LITFA group.

Denote by S the set of non-zero divisors of the ring $\mathcal{N}(\Gamma)$. By [Re98, Prop.2.8] (see also [Lü02b, Thm.8.22]) the pair $(\mathcal{N}(\Gamma), S)$ satisfies the right Ore condition. The ring $\mathcal{U}(\Gamma) := \mathcal{N}(\Gamma)S^{-1}$ is called the *algebra of operators affiliated to $\mathcal{N}(\Gamma)$* . For any $\mathcal{U}(\Gamma)$ -module \mathcal{M} we also have a dimension $\dim_{\mathcal{U}(\Gamma)}(\mathcal{M})$. By [Lü02b, Thm.8.31] we have

$$b_i^{(2)}(M, \alpha) = \dim_{\mathcal{U}(\Gamma)}(H_i(C_*^{sing}(M_\Gamma) \otimes_{\mathbb{Z}[\Gamma]} \mathcal{U}(\Gamma))).$$

We collect below some properties of LITFA groups (see [FLM09, Thm.2.2] and the references therein):

Theorem 2.4. *Let Γ be a LITFA group.*

- (1) *All non-zero elements in $\mathbb{Z}[\Gamma]$ are non-zero divisors in $\mathcal{N}(\Gamma)$.*
- (2) *$\mathbb{Z}[\Gamma]$ is an Ore domain and embeds in its classical right ring of quotients $\mathbb{K}(\Gamma)$, a skew-field.*
- (3) *$\mathbb{K}(\Gamma)$ is flat over $\mathbb{Z}[\Gamma]$.*
- (4) *There exists a monomorphism $\mathbb{K}(\Gamma) \rightarrow \mathcal{U}(\Gamma)$ which makes the following diagram commute*

$$\begin{array}{ccc} \mathbb{Z}[\Gamma] & \longrightarrow & \mathbb{K}(\Gamma) \\ & \searrow & \downarrow \\ & & \mathcal{U}(\Gamma). \end{array}$$

Remark. Since for a LITFA group Γ , $\mathbb{K}(\Gamma)$ is a skew-field, it follows that any $\mathbb{K}(\Gamma)$ -module is free. In particular, $\mathcal{U}(\Gamma)$ is flat as a $\mathbb{K}(\Gamma)$ -module.

The following result of [FLM09] relates L^2 -Betti numbers to ranks of modules over skew fields.

Proposition 2.5. ([FLM09, Prop.2.3]) *Let $\alpha : \pi_1(M) \rightarrow \Gamma$ be an epimorphism to a LITFA group Γ . Then*

$$b_i^{(2)}(M, \alpha) = \dim_{\mathbb{K}(\Gamma)}(H_i(M; \mathbb{K}(\Gamma))).$$

A group Γ is called *poly-torsion-free-abelian* (PTFA) if there exists a normal series

$$1 = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_{n-1} \subset \Gamma_n = \Gamma$$

such that Γ_i/Γ_{i-1} is torsion free abelian. It is easy to see that PTFA groups are LITFA. Note that the quotients $\pi/\pi_r^{(k)}$ of a group by terms in the rational derived series are PTFA (cf. [Co04, Ha05]).

We now recall the definition of the Cochran–Harvey invariants (which in the context of complex algebraic geometry were first studied in [LM06], for plane curve complements). Let X be a hypersurface in \mathbb{C}^n , with complement $M_X := \mathbb{C}^n \setminus X$. Furthermore let $\alpha : \pi_1(M_X) \rightarrow \Gamma$ be an *admissible* epimorphism to a LITFA group. Recall that “admissible” means that there exists a map $\tilde{\phi} : \Gamma \rightarrow \mathbb{Z}$ such that the following diagram commutes

$$\begin{array}{ccc} \pi_1(M_X) & \xrightarrow{\alpha} & \Gamma \\ & \searrow \phi & \swarrow \tilde{\phi} \\ & \mathbb{Z} & \end{array}$$

where $\phi : \pi_1(M_X) \rightarrow \mathbb{Z}$ is the total linking homomorphism. Denote by \widetilde{M}_X the infinite cyclic cover of M_X defined by the kernel of the total linking number homomorphism ϕ . Let $\tilde{\Gamma}$ be the kernel of $\tilde{\phi} : \Gamma \rightarrow \mathbb{Z}$ and denote the induced homomorphism $\pi_1(\widetilde{M}_X) \rightarrow \tilde{\Gamma}$ by $\tilde{\alpha}$.

Now consider the homomorphism $\pi_1(M_X) \rightarrow \pi_1(M_X)/\pi_1(M_X)_r^{(m+1)} =: \Gamma_m$. It is easy to see that this homomorphism is admissible. As in [LM06] we now define

$$\delta_{i,m}(X) = \dim_{\mathbb{K}(\tilde{\Gamma}_m)}(H_i(\widetilde{M}_X; \mathbb{K}(\tilde{\Gamma}_m))).$$

The following result, which is an immediate corollary to Prop.2.5, shows that the L^2 –Betti numbers of \widetilde{M}_X can be viewed as a generalization of the Cochran–Harvey invariants of affine hypersurface complements.

Theorem 2.6. *Let $X \subset \mathbb{C}^n$ be an affine hypersurface with complement M_X , and let $\alpha : \pi_1(M_X) \rightarrow \Gamma$ be an admissible epimorphism to a LITFA group. Then, in the above notations, we have*

$$\dim_{\mathbb{K}(\tilde{\Gamma})}(H_i(\widetilde{M}_X; \mathbb{K}(\tilde{\Gamma}))) = b_i^{(2)}(\widetilde{M}_X, \tilde{\alpha} : \pi_1(\widetilde{M}_X) \rightarrow \tilde{\Gamma}).$$

3. VANISHING OF L^2 -BETTI NUMBERS OF HYPERSURFACE COMPLEMENTS

Let X be a reduced hypersurface in \mathbb{C}^n ($n \geq 2$), defined by the equation $f = f_1 \cdots f_s = 0$, where f_i are the irreducible factors of f , and let $X_i = \{f_i = 0\}$ denote the irreducible components of X . Embed \mathbb{C}^n in \mathbb{CP}^n by adding the hyperplane at infinity, H , and let \bar{X} be the projective hypersurface in \mathbb{CP}^n defined by the homogenization f^h of f . Let M_X denote the affine hypersurface complement

$$M_X := \mathbb{C}^n \setminus X.$$

Alternatively, M_X can be regarded as the complement in \mathbb{CP}^n of the divisor $\bar{X} \cup H$. Then $H_1(M_X)$ is free abelian, generated by the meridian loops γ_i about the non-singular part of each irreducible component X_i , for $i = 1, \dots, s$ (cf. [Di92], (4.1.3), (4.1.4)).

Since M_X is a n -dimensional affine variety, it has the homotopy type of a finite CW-complex of dimension n (e.g., see (cf. [Di92], (1.6.7), (1.6.8)). Hence

$$(4) \quad b_i^{(2)}(M_X, \alpha) = 0, \quad \text{for all } i > n.$$

Let us now recall our notations. We start with an admissible epimorphism $\alpha : \pi_1(M_X) \rightarrow \Gamma$ to a group Γ , and consider the induced epimorphism $\tilde{\alpha} : \pi_1(\tilde{M}_X) \rightarrow \tilde{\Gamma}$, where $\tilde{\Gamma} := \text{Ker}(\tilde{\phi} : \Gamma \rightarrow \mathbb{Z})$, and \tilde{M}_X is the infinite cyclic cover of M_X defined by the total linking number homomorphism ϕ . As already noted in the introduction, ϕ coincides with the homomorphism $\pi_1(M_X) \rightarrow \pi_1(\mathbb{C}^*) = \mathbb{Z}$ induced by the polynomial map f (cf. [Di92, p.76-77]). The admissibility assumption implies that the Γ -cover of M_X defined by α factors through the infinite cyclic cover \tilde{M}_X .

We begin our investigation of L^2 -Betti numbers of hypersurface complements with the following special case:

Proposition 3.1. Let X be an affine hypersurface defined by a weighted homogeneous polynomial $f : \mathbb{C}^n \rightarrow \mathbb{C}$. Then

- (1) *All L^2 -Betti numbers $b_i^{(2)}(M_X, \alpha)$ of the complement M_X vanish.*
- (2) *All L^2 -Betti numbers $b_i^{(2)}(\tilde{M}_X, \tilde{\alpha})$ of the infinite cyclic cover \tilde{M}_X are finite, and $b_i^{(2)}(\tilde{M}_X, \tilde{\alpha}) = 0$ for $i \geq n$.*

Proof. Since the defining polynomial f of X is weighted homogeneous, there exist a global *Milnor fibration* (e.g., see [Mi68] or [Di92], (3.1.12)):

$$F = \{f = 1\} \hookrightarrow M_X = \mathbb{C}^n \setminus X \xrightarrow{f} \mathbb{C}^*.$$

Moreover, the fiber F has the homotopy type of a finite CW-complex of dimension $n-1$, and F is $(n-s-2)$ -connected, where s is the dimension of the singular locus of the hypersurface singularity germ $(X, 0)$. In particular, since X is reduced, F is connected. The vanishing of L^2 -Betti numbers $b_i^{(2)}(M_X, \alpha)$ of the complement follows now from Lemma 2.3.

Note the Milnor fiber F is homotopy equivalent to the infinite cyclic cover \tilde{M}_X of M_X corresponding to the kernel of the total linking number homomorphism. It follows that \tilde{M}_X has the homotopy type of a *finite* CW complex of dimension $n-1$, so the claim about the finiteness of the L^2 -Betti numbers of \tilde{M}_X follows readily. \square

Example. If X is a *central* hyperplane arrangement in \mathbb{C}^n (i.e., all hyperplanes pass through the origin), then Prop.3.1 yields that all L^2 -Betti numbers $b_i^{(2)}(M_X, \alpha)$ of the complement M_X vanish; this fact also follows from [DJL07].

Affine hypersurfaces defined by homogeneous polynomials are basic examples of *hypersurfaces in general position at infinity*, i.e., hypersurfaces $X \subset \mathbb{C}^n$ for which the hyperplane at infinity H is transversal in a stratified sense to the projective completion $\bar{X} \subset \mathbb{CP}^n$. In [Ma06], the author showed that for affine hypersurfaces X in general position at infinity the ordinary Betti numbers $b_i(\tilde{M}_X)$ of the infinite cyclic cover \tilde{M}_X are finite for all $0 \leq i \leq n-1$. In this paper we give a non-commutative generalization of this fact. We begin with the following comparison result.

Theorem 3.2. Let X be a hypersurface in \mathbb{C}^n , and let S^∞ be a $(2n-1)$ -sphere in \mathbb{C}^n of a sufficiently large radius (that is, the boundary of a small tubular neighborhood in \mathbb{CP}^n of the hyperplane H at infinity). Denote by $X^\infty = S^\infty \cap X$ the link of X at infinity, and by $M_X^\infty = S^\infty - X^\infty$ its complement in S^∞ . Let α^∞ be the composition map

$$\pi_1(M_X^\infty) \rightarrow \pi_1(M_X) \rightarrow \Gamma.$$

Denote by \widetilde{M}_X^∞ the infinite cyclic cover of M_X^∞ defined by the composition

$$\pi_1(M_X^\infty) \rightarrow \pi_1(M_X) \xrightarrow{\phi} \mathbb{Z},$$

and let $\widetilde{\alpha}^\infty : \pi_1(\widetilde{M}_X^\infty) \rightarrow \widetilde{\Gamma}$ be the induced homomorphism to $\widetilde{\Gamma} := \text{Ker}\{\widetilde{\phi} : \Gamma \rightarrow \mathbb{Z}\}$. Finally, let $b_i^{(2)}(M_X^\infty; \alpha^\infty)$ and $b_i^{(2)}(\widetilde{M}_X^\infty; \widetilde{\alpha}^\infty)$ be the L^2 -Betti numbers of M_X^∞ and \widetilde{M}_X^∞ , respectively.

Then for all $i \leq n - 1$ we have the inequalities

$$(5) \quad b_i^{(2)}(\widetilde{M}_X, \widetilde{\alpha}) \leq b_i^{(2)}(\widetilde{M}_X^\infty; \widetilde{\alpha}^\infty)$$

and

$$(6) \quad b_i^{(2)}(M_X, \alpha) \leq b_i^{(2)}(M_X^\infty; \alpha^\infty),$$

with equalities in (5) and (6) if $i < n - 1$.

Proof. First, it is clear that α^∞ is an admissible map. Next, note that M_X^∞ is homotopy equivalent to $T(H) \setminus \bar{X} \cup H$, where $T(H)$ is the tubular neighborhood of H in \mathbb{CP}^n for which S^∞ is the boundary. Then a classical argument based on the Lefschetz hyperplane theorem yields that the homomorphism $\pi_i(M_X^\infty) \rightarrow \pi_i(M_X)$ is an isomorphism for $i < n - 1$ and it is surjective for $i = n - 1$; see [DL06][Section 4.1] for more details. In particular, α^∞ is an epimorphism, as is the composite homomorphism $\pi_1(M_X^\infty) \rightarrow \mathbb{Z}$.

From the above considerations, it follows that

$$(7) \quad \pi_i(M_X, M_X^\infty) = 0, \text{ for all } i \leq n - 1,$$

hence M_X has the homotopy type of a complex obtained from M_X^∞ by adding cells of dimension $\geq n$. So the same is true for any covering, and in particular for the corresponding Γ -coverings. So the group homomorphisms

$$(8) \quad H_i((M_X^\infty)_\Gamma; \mathbb{Z}) \rightarrow H_i((M_X)_\Gamma; \mathbb{Z})$$

are isomorphisms if $i < n - 1$ and surjective for $i = n - 1$. Since these homomorphisms are induced by an embedding map, they are in fact homomorphisms of $\mathbb{Z}\Gamma$ -modules. The (in)equalities in (6) follow now from Lemma 2.2.

Next note that the Γ -cover $(M_X)_\Gamma$ of M_X is a $\widetilde{\Gamma}$ -cover of the infinite cyclic cover \widetilde{M}_X . Similar considerations apply to the covers of M_X^∞ . So (8) can be restated as saying that the group homomorphisms

$$(9) \quad H_i((\widetilde{M}_X^\infty)_{\widetilde{\Gamma}}; \mathbb{Z}) \rightarrow H_i((\widetilde{M}_X)_{\widetilde{\Gamma}}; \mathbb{Z})$$

are isomorphisms if $i < n - 1$ and surjective for $i = n - 1$. And, as before, these are in fact homomorphisms of $\mathbb{Z}\widetilde{\Gamma}$ -modules. Another application of Lemma 2.2 yields the (in)equalities of (5). □

In the next result, we restrict our attention to the case of hypersurfaces in general position at infinity. As it was already observed in a sequence of papers, e.g., see [DL06, DM07, Ma06], such hypersurfaces behave much like weighted homogeneous hypersurfaces up to homological degree $n - 1$.

Theorem 3.3. Assume that the affine hypersurface $X \subset \mathbb{C}^n$ is in general position at infinity, i.e., the hyperplane at infinity H is transversal in the stratified sense to the projective completion \bar{X} . Then

- (1) The L^2 -Betti numbers $b_i^{(2)}(\widetilde{M}_X, \widetilde{\alpha})$ of the infinite cyclic cover \widetilde{M}_X are finite for all $0 \leq i \leq n-1$. In particular, the Cochran-Harvey higher-order degrees $\delta_{i,m}(X)$ are finite for $0 \leq i \leq n-1$ and all integers m .
- (2) The L^2 -Betti numbers of the complement M_X are computed by

$$b_i^{(2)}(M_X, \alpha) = \begin{cases} 0, & \text{for } i \neq n, \\ (-1)^n \chi(M_X), & \text{for } i = n. \end{cases}$$

In particular,

$$(-1)^n \cdot \chi(M_X) \geq 0.$$

Proof. For the first part of the theorem, by Thm.3.2 it suffices to show that the L^2 -Betti numbers $b_i^{(2)}(\widetilde{M}_X^\infty; \widetilde{\alpha}^\infty)$ are finite for all $0 \leq i \leq n-1$. (This was proved in [FLM09] for $n = 2$.) Note that since \bar{X} is transversal to H , the space M_X^∞ is a circle fibration over $H \setminus \bar{X} \cap H$ which is homotopy equivalent to the complement in \mathbb{C}^n to the affine cone over the projective hypersurface $\bar{X} \cap H \subset H = \mathbb{CP}^{n-1}$ (for a similar argument see [DL06], Section 4.1). Let $\{h = 0\}$ be the polynomial defining $\bar{X} \cap H$ in H . Then the infinite cyclic cover \widetilde{M}_X^∞ of M_X^∞ is homotopy equivalent to the Milnor fiber $\{h = 1\}$ of the (homogeneous) hypersurface singularity at the origin defined by h . In particular, \widetilde{M}_X^∞ has the homotopy type of a finite CW-complex. So the claim about the finiteness of $b_i^{(2)}(\widetilde{M}_X^\infty; \widetilde{\alpha}^\infty)$ follows now from definition. Similarly, the finiteness of the Cochran-Harvey higher-order degrees $\delta_{i,m}(X)$ in the relevant range follows from the considerations of Section 2.3, where these degrees are realized as L^2 -Betti numbers of the infinite cyclic cover \widetilde{M}_X .

For the second part of the theorem, note that by the above considerations M_X^∞ is homotopic to the total space of a fibration over S^1 , namely the Milnor fibration at the origin corresponding to the homogeneous polynomial h . So, by Lemma 2.3 we obtain that all L^2 -Betti numbers $b_i^{(2)}(M_X^\infty; \alpha^\infty)$ vanish. The claim about the L^2 -Betti numbers of M_X now follows from the inequalities (6) of Thm.3.2, together with Lem.2.1(3). □

Remark. If Γ is a LITFA group as in Section 2.3, then the L^2 -Betti numbers are determined by ranks of homology modules over skew-fields. In this case, the flatness of certain rings involved shows that the finiteness of the L^2 -Betti number $b_i^{(2)}(\widetilde{M}_X, \widetilde{\alpha})$ of the infinite cyclic cover is equivalent to the vanishing of the L^2 -Betti number $b_i^{(2)}(M_X, \alpha)$ of the complement.

REFERENCES

- [Co04] T. Cochran, *Noncommutative knot theory*, *Algebr. Geom. Topol.* **4** (2004), 347–398.
- [DJL07] M. W. Davis, T. Januszkiewicz, I. J. Leary, *The l^2 -cohomology of hyperplane complements*, *Groups Geom. Dyn.* **1** (2007), no. 3, 301–309.
- [Di92] A. Dimca, *Singularities and topology of hypersurfaces*, Universitext, Springer-Verlag, New York (1992).
- [DL06] A. Dimca, A. Libgober, *Regular functions transversal at infinity*, *Tohoku Math. J. (2)* **58** (2006), no. 4, 549–564.
- [DM07] A. Dimca, L. Maxim, *Multivariable Alexander invariants of hypersurface complements*, *Trans. Amer. Math. Soc.* **359** (2007), 3505–3528.
- [FLM09] S. Friedl, C. Leidy, L. Maxim, *L^2 -Betti numbers of plane algebraic curves*, *Michigan Math. J.* **58** (2009), no. 2, 411–421.
- [Ha05] S. Harvey, *Higher-order polynomial invariants of 3-manifolds giving lower bounds for the Thurston norm*, *Topology* **44** (2005), 895–945.

- [JZ00] J. Jost, K.; Zuo, *Vanishing theorems for L^2 -cohomology on infinite coverings of compact Kähler manifolds and applications in algebraic geometry*, Comm. Anal. Geom. **8** (2000), no. 1, 1–30.
- [LM06] C. Leidy, L. Maxim, *Higher-order Alexander invariants of plane algebraic curves*, IMRN, Volume **2006** (2006), Article ID 12976, 23 pages.
- [LM07] C. Leidy, L. Maxim, *Obstructions on fundamental groups of plane curve complements*, in “Real and Complex Singularities”, Contemp. Math. **459** (2008), pp. 117–130.
- [Lü02a] W. Lück, *L^2 -invariants of regular coverings of compact manifolds and CW-complexes*, Handbook of geometric topology, North-Holland, Amsterdam, 2002, pp. 735–817.
- [Lü02b] W. Lück, *L^2 -invariants: Theory and Applications to Geometry and K-Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, **44**. Springer-Verlag, Berlin, 2002.
- [Ma06] L. Maxim, *Intersection homology and Alexander modules of hypersurface complements*, Comm. Math. Helv. **81** (1), 123–155, 2006.
- [Mi68] J. Milnor, *Singular points of complex hypersurfaces*, Annals of Mathematics Studies, No. **61**, Princeton University Press, Princeton, N.J. (1968)
- [Re98] H. Reich, *Group von Neumann algebras and related algebras*, Thesis, University of Göttingen (1998)

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